

Preprint ITP-96-16E

hep-th/9703037

Duality of the 2D Nonhomogeneous Ising Model on the Torus

A.I. Bugrij ¹, V.N. Shadura*Bogolyubov Institute for Theoretical Physics**252143 Kiev-143, Ukraine*

Abstract

Duality relations for the 2D nonhomogeneous Ising model on the finite square lattice wrapped on the torus are obtained. The partition function of the model on the dual lattice with arbitrary combinations of the periodical and antiperiodical boundary conditions along the cycles of the torus is expressed through some specific combination of the partition functions of the model on the original lattice with corresponding boundary conditions. It is shown that the structure of the duality relations is connected with the topological peculiarities of the dual transformation of the model on the torus.

PACS numbers: 05.50.+q

¹E-mail address: abugrijluk.apc.org

The duality relation for the two-dimensional Ising model was discovered by Kramers and Wannier [1]. In their work, Kramers and Wannier showed the correspondence between the partition function of the model in low-temperature phase and the partition function of the model on the dual lattice in high-temperature phase and vice versa:

$$(\cosh 2\widetilde{K})^{-N} \widetilde{Z}(\widetilde{K}) = (\cosh 2K)^{-N} Z(K) \quad (1)$$

$$\sinh 2K \cdot \sinh 2\widetilde{K} = 1.$$

Using this self-duality property, the critical temperature of the 2D Ising model was determined [1] before Onsager had obtained the exact solution [2].

Kadanoff and Ceva [3] generalized the Kramers-Wannier duality relation (1) to the nonhomogeneous case (the coupling constants are arbitrary functions of lattice site coordinates) with spherical boundary conditions

$$\prod_{\widetilde{r}, \mu} (\cosh 2\widetilde{K}_\mu(\widetilde{r}))^{-1/2} \widetilde{Z}[\widetilde{K}] = \prod_{r, \mu} (\cosh 2K_\mu(r))^{-1/2} Z[K], \quad (2)$$

$$\sinh 2K_x(r) \cdot \sinh 2\widetilde{K}_y(\widetilde{r}) = 1, \quad \sinh 2K_y(r) \cdot \sinh 2\widetilde{K}_x(\widetilde{r}) = 1. \quad (3)$$

Here $\mu = x, y$ and r, \widetilde{r} , $K_\mu(r)$, $\widetilde{K}_\mu(\widetilde{r})$ are coordinates and coupling constants on the original and dual lattices respectively. Since the Kadanoff-Ceva relation (2) defines the connection between functionals, this relation is very useful for analysis of the thermodynamic phases of the model. Thus, for example, this relation allows one to define correctly the disorder parameter μ , to obtain the duality relation connecting correlation functions on the original and dual lattices, to define "mixed" correlation functions $\langle \sigma(r_i) \dots \sigma(r_j) \mu(r_k) \dots \mu(r_l) \rangle$ and so on (see Ref. [3]).

As was already mentioned in Ref. [1,3], relations (1) and (2) can not be understood literally. So, for example, using the method of comparing high- and low-temperature expansions for deriving the duality relation (1) in the case of the periodical boundary conditions, it is hard to take into account and to compare the graphs wrapping up the torus. In fact Eq. (1) is correct in the thermodynamic limit (for the specific free energy). However for the nonhomogeneous case the procedure of thermodynamic limit is rather ambiguous. In Ref. [3] the duality relation (3) was obtained for spherical (nonphysical for the square lattice) boundary conditions.

Since duality is a popular method of nonperturbative investigation in field theory and statistical mechanics (for review see Ref. [4]), it is important to formulate a duality transformation for finite systems. Recently, we have suggested [5,6] the exact duality relations for the nonhomogeneous Ising model on a finite square lattice of size $N = n \times m$ wrapped on the torus:

$$\prod_{\widetilde{r}, \mu} (\cosh 2\widetilde{K}_\mu(\widetilde{r}))^{-1/2} \widetilde{\mathbf{Z}}[\widetilde{K}] = \prod_{r, \mu} (\cosh 2K_\mu(r))^{-1/2} \widehat{T} \mathbf{Z}[K]. \quad (4)$$

Here

$$\hat{T} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad \hat{T}^2 = 1. \quad (5)$$

and components of the four-vector $(\tilde{\mathbf{Z}}[\tilde{K}]$ for the dual lattice)

$$\mathbf{Z}[K] = (Z^{(p,p)}, Z^{(p,a)}, Z^{(a,p)}, Z^{(a,a)}),$$

are partition functions $Z^{(\alpha,\beta)}[K]$ ($\alpha, \beta = a, p$) of the Ising model with corresponding combinations of the periodical (p) and antiperiodical (a) boundary conditions along the horizontal X and vertical Y axes:

$$Z^{(\alpha,\beta)}[K] = \sum_{[\sigma]} e^{-\beta H^{(\alpha,\beta)}[K,\sigma]}, \quad (6)$$

$$-\beta H^{(\alpha,\beta)}[K,\sigma] = \sum_r \sigma(r) (K_x(r) \nabla_x^\alpha + K_y(r) \nabla_y^\beta) \sigma(r), \quad (7)$$

where $r = (x, y)$ denotes the site coordinates on the square lattice of size $N = n \times m$, $x = 1, \dots, n$, $y = 1, \dots, m$; $\sigma(r) = \pm 1$; $K_x(r)$ and $K_y(r)$ are the coupling constants along the horizontal X and vertical Y axes respectively. The one-step shift operators ∇_x , ∇_y act on $\sigma(r)$ in the following way

$$\nabla_x \sigma(r) = \sigma(r + \hat{x}), \quad \nabla_y \sigma(r) = \sigma(r + \hat{y}),$$

where \hat{x} , \hat{y} are the unit vectors along the horizontal and vertical axes. For the periodical (antiperiodical) boundary conditions along X and Y axes we have

$$\nabla_x^{p(a)} \sigma(n, y) = +(-) \sigma(1, y), \quad \nabla_y^{p(a)} \sigma(x, m) = +(-) \sigma(x, 1).$$

We denote site coordinates, functions and functionals on the dual lattice by "tilde" : \tilde{r} , $\tilde{\sigma}(\tilde{r})$, $\tilde{K}_\mu(\tilde{r})$, $\tilde{H}[\tilde{K}, \tilde{\sigma}]$, $\tilde{Z}[\tilde{K}]$, A site coordinate on the dual lattice coincides with a coordinate of the plaquet center on the original lattice:

$$\tilde{r} = r + (\hat{x} + \hat{y})/2. \quad (8)$$

In Ref. [5] the duality relation (4) was proved for homogeneous and weakly nonhomogeneous distributions of the coupling constants. We also have checked the duality relation (4) for lattices of small sizes by direct calculation on the computer. As a corollary of Eq. (4), we obtained [5,6] the duality relations for the two-point correlation function on the torus, for the partition functions of the 2D Ising model with magnetic fields applied to the boundaries and the 2D Ising model with free, fixed and mixed boundary conditions.

In this Letter we would like to propose a simple way of the derivation of the duality relation (4). For the derivation it is convenient to use the representation of Hamiltonian (7) in terms of Hamiltonian $H_D^{(\alpha,\beta)}$ of the Ising model with the magnetic dislocation corresponding to the boundary conditions (α, β) and the periodical boundary conditions for ∇_μ :

$$H_D^{(\alpha,\beta)}[K^{(\alpha,\beta)}, \sigma] = \sum_{r,\mu} \sigma(r) K_\mu^{(\alpha,\beta)}(r) \nabla_\mu^p \sigma(r), \quad \mu = x, y.$$

Here the coupling constants configurations $[K^{(\alpha,\beta)}]$ define the following magnetic dislocations:

(i) For the Hamiltonian $H_D^{(p,a)}$:

$$D_X = \begin{cases} K_x^{(p,a)}(r') = K_x(r'), \\ K_y^{(p,a)}(r') = -K_y(r'), \text{ if } r' \in B_X^{(m)}; \\ K_\mu^{(p,a)}(r) = K_\mu(r), \text{ if } r \notin B_X^{(m)}; \end{cases}$$

where we introduced the denotation $B_X^{(i)}$ for the following boundary cycles on the torus

$$B_X^{(i)} = \{(x, i), x = 1, \dots, n\}, \quad i = 1, m,$$

(ii) For the Hamiltonian $H_D^{(a,p)}$:

$$D_Y = \begin{cases} K_x^{(a,p)}(r') = -K_x(r'), \\ K_y^{(a,p)}(r') = K_y(r'), \text{ if } r' \in B_Y^{(n)}; \\ K_\mu^{(a,p)}(r) = K_\mu(r), \text{ if } r \notin B_Y^{(n)}; \end{cases}$$

where the path $B_Y^{(i)}$ denotes the other boundary cycles on the torus

$$B_Y^{(i)} = \{(i, y), y = 1, \dots, m\}, \quad i = 1, n,$$

(iii) For the Hamiltonian $H_D^{(a,a)}$ we have $D_{X,Y} = D_X \cup D_Y$.

It is evident that $H_D^{(p,p)}$ has not magnetic dislocation. Nevertheless, let us denote configuration of the coupling constants in this case as D_0 and call coupling constants configurations $D_0, D_X, D_Y, D_{X,Y}$ as the basic magnetic dislocations for corresponding $H_D^{(\alpha,\beta)}$.

Note that the partition function (6) is invariant with respect to the Z_2 local gauge transformation $\hat{U}[\tau]$ [7,8]:

$$\hat{U}[\tau][K, \sigma] = [K', \sigma'] = [\{\bar{K}_\mu(r) = \tau(r)K_\mu(r)\tau(r + \hat{\mu})\}, \{\sigma'(r) = \tau(r)\sigma(r)\}], \quad (9)$$

where $\tau(r) = \pm 1$. Let us apply arbitrary gauge transformation to the partition function $Z^{(\alpha,\beta)}$

$$\hat{U}[\tau] \sum_{[\sigma]} \exp\{-\beta H_D^{(\alpha,\beta)}[K, \sigma]\} = \sum_{[\sigma']} \exp\{-\beta H_D^{(\alpha,\beta)}[K', \sigma']\} = Z^{(\alpha,\beta)}[K'] = Z^{(\alpha,\beta)}[K].$$

Here new coupling constants configuration $[K']$ can contain both the deformation of the corresponding basic magnetic dislocations and new closed magnetic dislocations. Let us denote by $\Omega^{(p,p)}, \Omega^{(p,a)}, \Omega^{(a,p)}, \Omega^{(a,a)}$ the classes of gauge equivalent configurations $[K']$ generated by the gauge transformations from the basic magnetic dislocations $D_0, D_X, D_Y, D_{X,Y}$ respectively. It is evident that these classes have not intersections because the basic magnetic dislocations are the homotopy-nonequivalent paths on the two-dimensional torus (two arbitrary coupling constants configurations from different classes $\Omega^{(\alpha,\beta)}$ can't be connected by the Z_2 gauge transformations).

For the dual transformation of the partition function (6) let us use the standard method [4] of the passage to dual spins. At the beginning we consider the dual transformation of $Z^{(p,p)}$:

$$\begin{aligned} Z^{(p,p)}[K] &= \sum_{[\sigma]} \exp\left[\sum_{r,\mu} \sigma(r) K_\mu(r) \nabla_\mu^p \sigma(r)\right] = \\ &= \sum_{[\sigma]} \prod_{r,\mu} \left[\sum_{l_\mu} P_{l_\mu(r)}(K_\mu(r)) (\sigma(r) \sigma(r + \hat{\mu}))^{l_\mu(r)} \right] = \\ &= \sum_{[l_\mu]} \sum_{[\sigma]} \prod_{r,\mu} P_{l_\mu(r)}(K_\mu(r)) \prod_r (\sigma(r))^{\psi(r)} = 2^N \sum_{[l_\mu]} \prod_{r,\mu} P_{l_\mu(r)}(K_\mu(r)) \prod_r \delta_2[\psi(r)] \end{aligned} \quad (10)$$

where

$$P_{l_\mu(r)}(K_\mu(r)) = \cosh(K_\mu(r)) \exp(l_\mu(r) \ln \tanh K_\mu(r)) \quad (11)$$

and

$$\psi(r) = l_x(r) + l_y(r) + l_x(r - \hat{x}) + l_y(r - \hat{y}), \quad l_\mu(r) = 0, 1. \quad (12)$$

Here $\delta_2(\psi)$ is a Kronecker δ -function *mod* 2 : it is zero if ψ is odd and one if ψ is even. Note that in Eq. (10) we have a product of the δ -functions with linking arguments (the same l_μ is contained in two δ -functions). In order to solve constraints generated by the product of δ -functions $l_\mu(r)$ is usually expressed through dual spins $\tilde{\sigma}(\tilde{r})$ [4]

$$l_\mu(r) = \frac{1}{2}(1 - \tilde{\sigma}(\tilde{r})\tilde{\sigma}(\tilde{r} - \nu)), \quad \mu \neq \nu. \quad (13)$$

Substituting Eq. (12) in Eq. (9) we obtain

$$\begin{aligned} Z^{(p,p)}[K] &= \frac{1}{2} \sum_{[\tilde{\sigma}]} \prod_{r,\mu} (\sinh 2K_\mu(r))^{\frac{1}{2}} \exp\left[\sum_{\tilde{r},\nu} \tilde{\sigma}(\tilde{r}) \tilde{K}_\nu(\tilde{r}) \nabla_\nu^p \tilde{\sigma}(\tilde{r})\right] = \\ &= \frac{1}{2} \prod_{r,\mu} (\cosh 2\tilde{K}_\nu(\tilde{r}))^{-1} \tilde{Z}^{(p,p)}[\tilde{K}], \end{aligned} \quad (14)$$

where the dual coupling constants $\tilde{K}_\nu(\tilde{r})$ are defined by (3) or

$$\tanh K_\mu(r) = e^{-2\tilde{K}_\nu(\tilde{r})}, \quad \mu \neq \nu. \quad (15)$$

To derive Eq. (14) we used identity

$$\frac{\cosh^2 2K_\mu(r)}{\sinh 2K_\mu(r)} = \frac{\cosh^2 2\widetilde{K}_\nu(\tilde{r})}{\sinh 2\widetilde{K}_\nu(\tilde{r})}, \quad \mu \neq \nu.$$

Since in Eq. (14) we sum over $[\tilde{\sigma}]$ (this counts each configuration $[l_\mu]$ twice because $[\tilde{\sigma}] \rightarrow [-\tilde{\sigma}]$ gives the same $[l_\mu]$), we must introduce factor $1/2$.

However it is not hard to note that we can construct many other solutions for l_μ for which the coupling constant configurations are connected with $[\widetilde{K}]$ in (14) by means of the gauge transformation $\widehat{U}[\tilde{\tau}]$ on the dual lattice (see Eq. (9)). Such configurations form a class $\widetilde{\Omega}^{p,p}$ of gauge-equivalent coupling constant configurations. By analogy with the Ising model on the original lattice wrapped on the torus we must expect the existence of solutions for l_μ which lead to homotopy-nonequivalent classes $\widetilde{\Omega}^{(p,p)}, \widetilde{\Omega}^{(p,a)}, \widetilde{\Omega}^{(a,p)}, \widetilde{\Omega}^{(a,a)}$ of dual coupling constants configurations. Really it is easy to write the solutions for l_μ which lead to the basic magnetic dislocations $\widetilde{D}_X, \widetilde{D}_Y, \widetilde{D}_{X,Y}$ in $\widetilde{\Omega}^{(p,a)}, \widetilde{\Omega}^{(a,p)}, \widetilde{\Omega}^{(a,a)}$ respectively. So, taking into account Eq. (8), we have for \widetilde{D}_X :

$$l_x(r) = \frac{1}{2}(1 + \tilde{\sigma}(\tilde{r})\tilde{\sigma}(\tilde{r} - \hat{y})), \quad (16)$$

$$l_y(r) = \frac{1}{2}(1 - \tilde{\sigma}(\tilde{r})\tilde{\sigma}(\tilde{r} - \hat{x})), \text{ if } r \in B_X^{(1)};$$

$$l_\mu(r) = \frac{1}{2}(1 - \tilde{\sigma}(\tilde{r})\tilde{\sigma}(\tilde{r} - \hat{\nu})), \text{ if } r \notin B_X^{(1)};$$

for \widetilde{D}_Y :

$$l_y(r) = \frac{1}{2}(1 + \tilde{\sigma}(\tilde{r})\tilde{\sigma}(\tilde{r} - \hat{x})), \quad (17)$$

$$l_x(r) = \frac{1}{2}(1 - \tilde{\sigma}(\tilde{r})\tilde{\sigma}(\tilde{r} - \hat{y})), \text{ if } r \in B_Y^{(1)};$$

$$l_\mu(r) = \frac{1}{2}(1 - \tilde{\sigma}(\tilde{r})\tilde{\sigma}(\tilde{r} - \nu)), \text{ if } r \notin B_Y^{(1)};$$

and for $\widetilde{D}_{X,Y}$:

$$l_x(r) = \frac{1}{2}(1 + \tilde{\sigma}(\tilde{r})\tilde{\sigma}(\tilde{r} - \hat{y})), \quad (18)$$

$$l_y(r) = \frac{1}{2}(1 - \tilde{\sigma}(\tilde{r})\tilde{\sigma}(\tilde{r} - \hat{x})), \text{ if } r \in B_X^{(1)};$$

$$l_y(r) = \frac{1}{2}(1 + \tilde{\sigma}(\tilde{r})\tilde{\sigma}(\tilde{r} - \hat{x})),$$

$$l_x(r) = \frac{1}{2}(1 - \tilde{\sigma}(\tilde{r})\tilde{\sigma}(\tilde{r} - \hat{y})), \text{ if } r \in B_Y^{(1)};$$

$$l_\mu(r) = \frac{1}{2}(1 - \tilde{\sigma}(\tilde{r})\tilde{\sigma}(\tilde{r} - \nu)), \text{ if } r \notin B_X^{(1)}, B_Y^{(1)}.$$

In Eqs. (16)-(18) $\mu \neq \nu$.

Since arbitrary solutions satisfying the product of δ -functions in Eq. (10) lead to the dual coupling constant configurations which contain the finite number of closed magnetic dislocations on the dual lattice, it is obvious that these configurations can be generated by the gauge transformations from the basic magnetic dislocations and occur in the corresponding class $\tilde{\Omega}^{(\alpha,\beta)}$. Then in Eqs. (10), (14) at transformation to the dual lattice we must sum over the homotopy-nonequivalent solutions (13), (16)-(18). Taking into account that basic magnetic dislocations $\tilde{D}_0, \tilde{D}_X, \tilde{D}_Y, \tilde{D}_{X,Y}$ lead to the partition functions with corresponding boundary conditions, we obtain:

$$\prod_{r,\mu} (\cosh 2K_\mu(r))^{-\frac{1}{2}} Z^{(p,p)}[K] = \frac{1}{2} \prod_{\tilde{r},\nu} (\cosh 2\tilde{K}_\nu(\tilde{r}))^{-\frac{1}{2}} \left(\tilde{Z}^{(p,p)}[\tilde{K}] + \tilde{Z}^{(p,a)}[\tilde{K}] + \tilde{Z}^{(a,p)}[\tilde{K}] + \tilde{Z}^{(a,a)}[\tilde{K}] \right). \quad (19)$$

Now, let us consider the dual transformation of partition function $Z^{(p,a)}$. As we have discussed above, it contains magnetic dislocation D_X . Again, doing the duality transformation, it is necessary to sum over the homotopy-nonequivalent solutions (13), (16)-(18). As a result we obtain the duality relations similar to Eq. (19) but with the minus signs before $\tilde{Z}^{(p,a)}[\tilde{K}]$ and $\tilde{Z}^{(a,a)}[\tilde{K}]$. Appearance of these signs is connected with the presence of magnetic dislocation D_X in $Z^{(p,a)}$. Let us show this. Considering the transformation to the dual lattice in the same way as in Eqs. (10), (14), we obtain the partition function $\tilde{Z}^{(p,p)}[\tilde{K}^{(p,p)}]$ with the following dislocation (see Ref. [3]):

$$\tilde{G}_X = \begin{cases} \tilde{K}_x^{(p,p)}(\tilde{r}') = \tilde{K}_x(\tilde{r}') + i\frac{\pi}{2}, \\ \tilde{K}_y^{(p,p)}(\tilde{r}') = \tilde{K}_y(\tilde{r}'), \text{ if } \tilde{r}' \in \tilde{B}_X^{(m)}; \\ \tilde{K}_\mu^{(p,p)}(\tilde{r}) = \tilde{K}_\mu(\tilde{r}), \text{ if } \tilde{r} \notin \tilde{B}_X^{(m)}; \end{cases}$$

where the path $\tilde{B}_X^{(m)}$ denotes the following boundary cycle on the dual torus

$$\tilde{B}_X^{(m)} = \{(\tilde{x}, m), \tilde{x} = 1, \dots, n\},$$

that is the dual transformation transforms dislocation D_X to dislocation \tilde{G}_X .

The partition function $\tilde{Z}^{(p,p)}[\tilde{K}^{(p,p)}]$ can be written through the following correlation function [3]

$$i^n < \tilde{\sigma}(1, m) \tilde{\sigma}(2, m) \tilde{\sigma}(2, m) \tilde{\sigma}(3, m) \dots \tilde{\sigma}(n, m) \tilde{\sigma}(n+1, m) > \tilde{Z}^{(p,p)}[\tilde{K}] = i^n \tilde{Z}^{(p,p)}[\tilde{K}],$$

where $\tilde{Z}^{(p,p)}[\tilde{K}]$ does not contain magnetic dislocations. In other hand, observing that except the dislocation \tilde{G}_X the partition functions $\tilde{Z}^{(p,a)}[\tilde{K}^{(p,a)}]$ and $\tilde{Z}^{(a,a)}[\tilde{K}^{(a,a)}]$ have magnetic dislocation \tilde{D}_y , we obtain, for example, for the last partition function

$$i^n < \tilde{\sigma}(1, m) \tilde{\sigma}(2, m) \tilde{\sigma}(2, m) \tilde{\sigma}(3, m) \dots \tilde{\sigma}(n, m) \tilde{\sigma}(n+1, m) > \tilde{Z}^{(a,a)}[\tilde{K}] = -i^n \tilde{Z}^{(a,a)}[\tilde{K}].$$

Here the minus sign appears because of the antiperiodical boundary conditions. Similarly the minus sign appears before $\tilde{Z}^{(p,a)}[\tilde{K}]$. Then, taking into account these signs, one gets

$$\prod_{r,\mu} (\cosh 2K_\mu(r))^{-\frac{1}{2}} Z^{(p,a)}[K] = \frac{1}{2} \prod_{\tilde{r},\nu} (\cosh 2\tilde{K}_\nu(\tilde{r}))^{-\frac{1}{2}} \left(\tilde{Z}^{(p,p)}[\tilde{K}] - \tilde{Z}^{(p,a)}[\tilde{K}] + \tilde{Z}^{(a,p)}[\tilde{K}] - \tilde{Z}^{(a,a)}[\tilde{K}] \right), \quad (20)$$

where the factor i^n is cancelled by the same factor appearing from the normalizing product $\prod \cosh^{1/2} \tilde{K}$. The same way, considering the dual transformation for $Z^{(a,p)}$ and $Z^{(a,a)}$, we obtain

$$\prod_{r,\mu} (\cosh 2K_\mu(r))^{-1/2} \mathbf{Z}[K] = \prod_{\tilde{r},\mu} (\cosh 2\tilde{K}_\mu(\tilde{r}))^{-1/2} \hat{T} \tilde{\mathbf{Z}}[\tilde{K}]. \quad (21)$$

Multiplying the right and the left sides of Eq. (21) by matrix \hat{T} (5), one obtains Eq. (4).

Thus the structure of duality relation (4) is connected with the topological peculiarities of the dual transformation of the Ising model on the torus. From here, one can assume that dual relations for other models with Z_2 symmetry, for example, the eight-vertex model, have the similar structure. The method of derivation of duality relation (4) suggested in this paper can be generalized for the two dimensional lattice models with Z_N symmetry. Results of this research will be published in the following paper.

The authors are indebted to A. Belavin and A. Morozov for continuous support and are grateful to M. Lashkevich and A. Mironov for useful discussions. One of the authors (V.S.) thanks A. Morozov for the hospitality and the excellent conditions at ITEP where this work has been begun. This work is partially supported by grant INTAS-93-1038.

Reference

1. H.A. Kramers, G.H. Wannier, Phys. Rev. **60**, 252 (1941).
2. L. Onsager, Phys. Rev. **65**, 117 (1944).
3. L.P. Kadanoff, H. Ceva, Phys. Rev. B **3**, 3918 (1971).
4. R. Savit, Rev. Mod. Phys. **52**, 453 (1980).
5. A.I. Bugrij, V.N. Shadura, Zh. Eksp. Teor. Fiz. **109**, 1024 (1996).
6. A.I. Bugrij, V.N. Shadura, Pis'ma Zh. Eksp. Teor. Fiz. **63**, 369 (1996).
7. G. Toulouse, Commun. Phys. **2**, 115 (1977).
8. E. Fradkin, V.A. Huberman, S.H. Shenker, Phys. Rev. B **18**, 4789 (1978).